YARIM CHEKSIZ INTERVALDA ISSIQLIK TARQALISH TENGLAMASI UCHUN FOKASNING UMUMLASHGAN ALMASHTIRISH USULI

Saparbayev Zafar Soatboyevich
Renessans ta'lim universiteti,
E-mail: saparbayev1603@gmail.com
Rajabov Azamat Zokirboyevich
Osiyo Xalqaro Universiteti

Annotatsiya: Ma'lumki tarmoqlangan sohalarda to'lqin tarqalishi va diffuziya jarayonlari ko'plab sohalarda uchraydi. Masalan, neyron birikmalarida nerv impulslarining tarqalishi, orterial qon aylanish sistemasida puls tarqalishi, elektr tarmoqlaridagi to'lqin jarayonlarini keltirish mumkin. Bunday jarayonlar odatda metrik graflarda differensial tenglamalar uchun boshlang'ich-chegaraviy masalalar yordamida modellashtiriladi.

Metrik graflarda differensial tenglamalar sohasini o'rganish nisbatan yangi soha hisoblanib, Asosan 1990 yillarga kelib grafda chiziqli Shredinger tenglamasi (Kvant grafi) o'rganila boshlangan. Keyinchalik esa graflarda boshqa turdagi tenglamalar 2010-yillarga kelib o'rganila boshlangan. Bulardan chiziqsiz Shredinger tenglamasi, Sin-Gardon tenglamasi, KdV tenglamalari bo'yicha olingan natijalarni keltirishimiz mumkin.

Ushbu maqolada chiziqli va ba'zi chiziqli bo'lmagan differensial tenglamalar uchun Fokasning umumlashgan almashtirish usuliga bag'ishlanadi.

Kalit so'zlar: Issiqlik tarqalish tenglamasi, Fokas metodi, metrik graflar, Grin formulasi, Furye almashtirishi.

Annotation: it is known that in branched areas, wave propagation and diffusion processes occur in many areas. Examples include the propagation of nerve impulses in neural connections, pulse propagation in the orterial circulatory system, and wave processes in electrical circuits. Such processes are typically modeled using initial-boundary issues for differential equations in metric graphs.

The study of the field of differential equations in metric graphs is a relatively new field, and mainly by the 1990s the linear Schrödinger equation (Quantum graph) began to be studied in a graph. Later, other types of equations in graphs began to be studied by the 2010s. From these we can derive the results obtained in terms of the lineless Schrödinger equation, The Sin-Gardon equation, the KDV equations.The study of the field of differential equations in metric graphs is a relatively new field, and mainly by the 1990s the linear Schrödinger equation (Quantum graph) began to be studied in a graph. Later, other types of equations in graphs began to be studied by the 2010s. From these we can derive the results

ISSN 3030-3907

obtained in terms of the lineless Schrödinger equation, The Sin-Gardon equation, the KDV equations.

This article is devoted to the generalized Fokas substitution method for linear and some nonlinear differential equations.

Keywords: heat dissipation equation, Phocas method, metric graphs, Green's formula, Fourier transform.

Аннотация: известно, что в разветвленных областях процессы распространения волн и диффузии происходят во многих областях. Например, можно привести распространение нервных импульсов в нейронных соединениях, распространение пульса в ортодонтической системе кровообращения, волновые процессы в электрических сетях. Такие процессы обычно моделируются на метрических графиках с использованием элементарно-граничных задач для дифференциальных уравнений.

Изучение области дифференциальных уравнений в метрических графах является относительно новой областью, и в основном линейное уравнение Шредингера (квантовый граф) начали изучать на графе примерно в 1990-х годах. Позже к 2010-м годам начали изучать другие типы уравнений на графиках. Из них можно вывести результаты, полученные по нелинейному уравнению Шредингера, уравнению Син-Гардона, уравнениям КДВ.Изучение области дифференциальных уравнений в графах является относительно новой областью, и в основном линейное уравнение Шредингера (квантовый граф) начали изучать на графе примерно в 1990-х годах. Позже к 2010-м годам начали изучать другие типы уравнений на графиках. Из них можно вывести результаты, полученные по нелинейному уравнению Шредингера, уравнению Син-Гардона, уравнениям КДВ.

Эта статья будет посвящена обобщенному методу подстановки Фокаса для линейных и некоторых нелинейных дифференциальных уравнений.

Ключевые слова: уравнение рассеивания тепла, метод Фокаса, метрические графики, формула Грина, подстановка Фурье.

Kirish.

So‘nggi yillarda metrik graflarda chiziqli va nochiziqli tenglamalar uchun boshlang‘ich-chegaraviy masalalarni yechish va bu yechimlarning xossalarini aniqlash bo‘yicha ilmiy izlanishlar olib borilmoqda. Nerv sistemalarida impuls tarqalishi jarayonini matematik modellashtirishning eng samarali usuli metrik grafda issiqlik tarqalish tenglamasi uchun maxsus boshlang‘ich-chegaraviy masalalardan foydalaniladi. Bunda, Fokasning umumlashgan almashtirish usulini metrik graflarda boshlang‘ich-chegaraviy masalalarni yechishga moslashtirish, metrik graflarda boshlang‘ich-

ISSN 3030-3907

chegaraviy masalalarni yechish usullarini ishlab chiqishga alohida e'tibor berilmoqda.

Ushbu ishi sodda metrik graflarda issiqliq tarqalish, Eyri tenglamalari uchun chegaraviy masalalarni qo‘yish va bu masalalarning yechimini Fokas usuli yordamida qurish, olingan natijalarning ba’zi tadbiqlarini tahlil qilishga bag‘ishlangan. Jahon miqyosida olib borilayotgan ko‘plab ilmiy-amaliy tadqiqotlar aksariyat hollarda xususiy hosilali differensial tenglamalar nazariyasi masalalariga keltiriladi. Xususiy hosilali differensial tenglamalar uchun boshlang‘ich-chegaraviy masalalar va kompleks o‘zgaruvchili funksiyalar nazariyasiga asoslangan metrik graflardagi masalalar biofizika, biotexnologiya, tibbiyot hamda ishlab chiqarish kabi sohalarning tadqiqot obyektidir. Tarmoqlangan strukturalarda kvant fizikasi va to‘lqin jarayonlarini modellashtirishda metrik graflarda aniqlangan issiqlik tarqalish va Shredinger tenglamalari uchun boshlang‘ich-chegaraviy masalalarni yechish muhim rol o‘ynaydi. Shu sababli metrik graflarda xususiy hosilali differentsial tenglamalar uchun boshlang‘ich-chegaraviy masalalarni yechish zamonaviy matematikada muhim ahamiyatga ega hisoblanadi.

Metrik graflarda boshlang‘ich-chegaraviy masalalalarni tadqiq qilish zarurati o‘tgan asrning 80-yillarning oxirlarida paydo bo‘lgan. Bunda Chexiyalik olimlar P.Eksner va P.Sebalar yulduzsimon ochiq graflarda Shredinger tenglamasi uchun chegaraviy masalalar qaralgan va olingan natijaning kvant zarrachasining tarmoqlangan sohada erkin harakatini o‘rganishga qaratilgan. Bu sohada hozirda mavjud ilmiy natijalar Z.A.Sobirov, M.I.Ahmedov, N.Uesker, D.Noya, D.Magnola, K.Seyfertlar tomonidan olingan. Chegaraviy masalalarni yechishning Fokas umumlashgan almashtirish usuli bugungi kunda yangi, rivojlanib kelayotgan usullardan biridir. Bu usul A.Fokas, D.Smit, N.Sheylslar tomonidan chekli va cheksiz intervallarda chiziqli va nochiziqli tenglamalar uchun qator boshlang‘ich va chegaraviy masalalar yechishga qo‘llanilgan.

Masalaning qo‘yilishi

Musbat yarim chiziqda berilgan issiqlik tarqalish tenglamasini ko‘rib chiqamiz:

$$u_t(x,t) = u_{xx}(x,t),\ x \in (0,t) > 0, \qquad (1)$$

$$u(x,0) = u_0(x), x \in (0), \qquad (2)$$

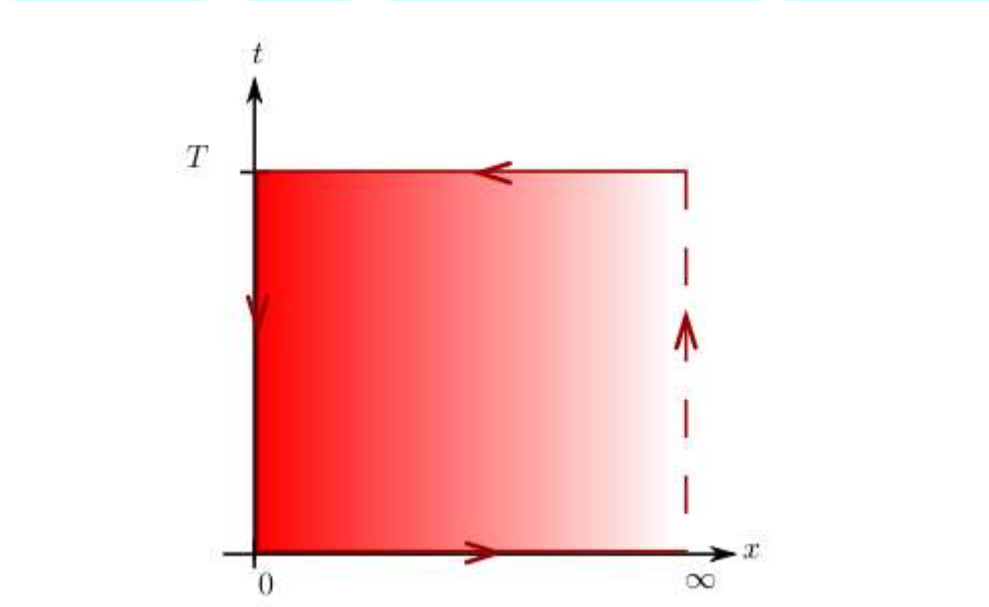
$$\lim_{x \to \Gamma} u(x,t) = 0. \qquad (3)$$

Dirixle chegaraviy shartlari berilgan deb faraz qilamiz. Ya’ni $u(0,t)$ ma’lum. Albatta bu muammo klassik usullar yordamida osonlikcha hal qilinadi. Biroq, bu oddiy muammoni Fokas usuli yordamida hal qilish masalasini ko‘rib chiqamiz. Buning uchun (1) tenglamani quyidagicha ko‘rinishda tasvirlab olamiz. Ya’ni,

ISSN 3030-3907

$$\left(e^{-ikx+w(k)t}u(x,t)\right)_t = \left(e^{-ikx+w(k)t}\left(u_x(x,t)+iku(x,t)\right)\right)_x, \qquad (4)$$

Bu yerda $w(k)=k^2$ dispersion munosabat deb ataladi. (4) tenglikka “lokal munosabat” deyiladi. Endi (4) munosabatga $(0,\Gamma)\times(0,t)$ o‘ng yarim tekislikda Grin formulasini qo‘llaymiz [1, 2].



1-rasm. Bitta yarim cheksiz intervalda Grin formulasini qo‘llash sohasi

$$\int_0^t\int_0^\Gamma \left(e^{-ikx+k^2s}u\right)_s - \left(e^{-ikx+k^2s}\left(u_x+iku\right)\right)_x dxds = 0$$

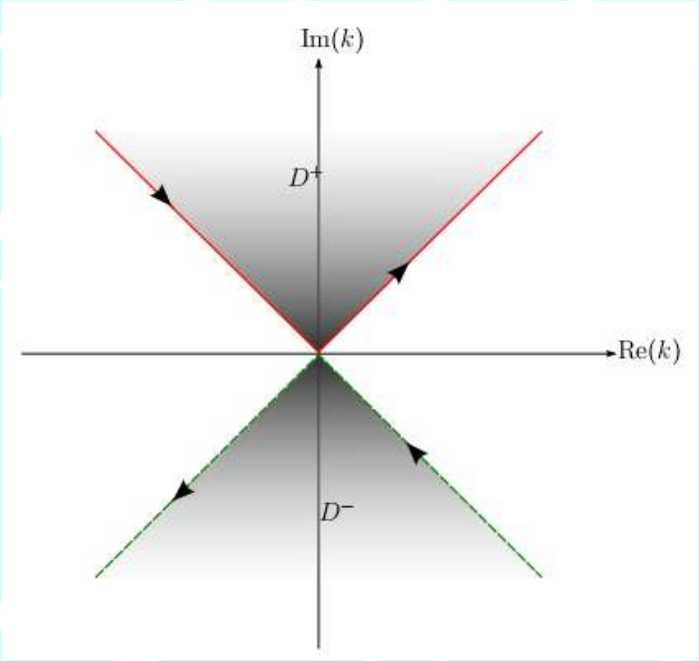
bundan esa

$$\int_0^\Gamma e^{-ikx}u_0(x)dx - \int_0^\Gamma e^{-ikx+k^2t}u(x,t)dx - \int_0^t e^{k^2s}\left(u_x(0,s)+iku(0,s)\right)ds = 0. \qquad (5)$$

J kompleks sonlar va $J^+ = \{z \in J : \mathrm{Im}\, k > 0\}$. Xuddi shuningdek, $J^- = \{z \in J : \mathrm{Im}\, k < 0\}$.

x yetarlicha katta bo‘lishi mumkinligi sababli, yuqoridagi integrallar yaqinlashuvchanligini ta’minlash uchun k ning mavhum qismini manfiy bo‘lishini talab qilamiz, ya’ni $k \in J^-$. Aytaylik $D = \{k \in J : \mathrm{Re}(k^2) < 0\} = D^+ \cup D^-$ soha bo‘lsin (2-rasm) [5, 6, 7].

ISSN 3030-3907



2-rasm. Issiqlik tenglamasi uchun D^+ va D^- sohalar k OJ uchun quyidagi belgilashlarni qaraymiz:

$$\hat{u}_0(k) = \int_0^{\infty} e^{-ikx} u_0(x)dx,$$

$$\hat{u}(k,t) = \int_0^{\infty} e^{-ikx} u(x,t)dx,$$

$$g_0(w,t) = \int_0^{t} e^{ws} u(0,s)ds,$$

$$g_1(w,t) = \int_0^{t} e^{ws} u_x(0,s)ds.$$

Ushbu belgilashlardan foydalanib,(5) tenglikdan global munosabat deb ataluvchi munosabatni hosil qilamiz [1] – [4]:

$$\hat{u}_0(k) - e^{k^2 t}\hat{u}(k,t) - g_1(w,t) - ikg_0(w,t) = 0. \qquad (6)$$

$w = k^2$ dispersion munosabat $k \to -k$ almashtirishga nisbatan invariant bo‘lganligi uchun $g_0(w,t), g_1(w,t)$ funksiyalar ham bu almashtirishga nisbatan invariant. Shu sababli (6) global munosabatdan quyidagini hosil qilishimiz mumkin [5]:

$$\hat{u}_0(-k) - e^{k^2 t}\hat{u}(-k,t) - g_1(w,t) + ikg_0(w,t) = 0. \qquad (7)$$

(7) tenglama k OJ$^+$ uchun o‘rinli.

(6) global munosabat uchun teskari Furye almashtirishini qo‘llab, masalaning yechimi qanday ko‘rinishdaligini topish mumkin, ya’ni

ISSN 3030-3907

$$u(x,t)=\frac{1}{2p}\int_{-\Gamma}^{\Gamma} e^{ikx-k^2t}\hat{u}_0(k)dk-\frac{1}{2p}\int_{-\Gamma}^{\Gamma} e^{ikx-k^2t}\left(g_1(w,t)+ikg_0(w,t)\right)dk, \qquad (8)$$

$x>0$ va $t>0$ uchun. (8) da birinchi integral haqida hamma narsa ma'lum. Ikkinchi integraldagi integral ostidagi ifodalarning $k\in J^+\setminus D^+$ uchun $k\to\Gamma$ da kamayib nolga intilishini e'tiborga olib, quyidagi integral ifodani hosil qilamiz:

$$u(x,t)=\frac{1}{2p}\int_{-\Gamma}^{\Gamma} e^{ikx-k^2t}\hat{u}_0(k)dk-\frac{1}{2p}\int_{\partial D^+} e^{ikx-k^2t}\left(g_1(w,t)+ikg_0(w,t)\right)dk. \qquad (9)$$

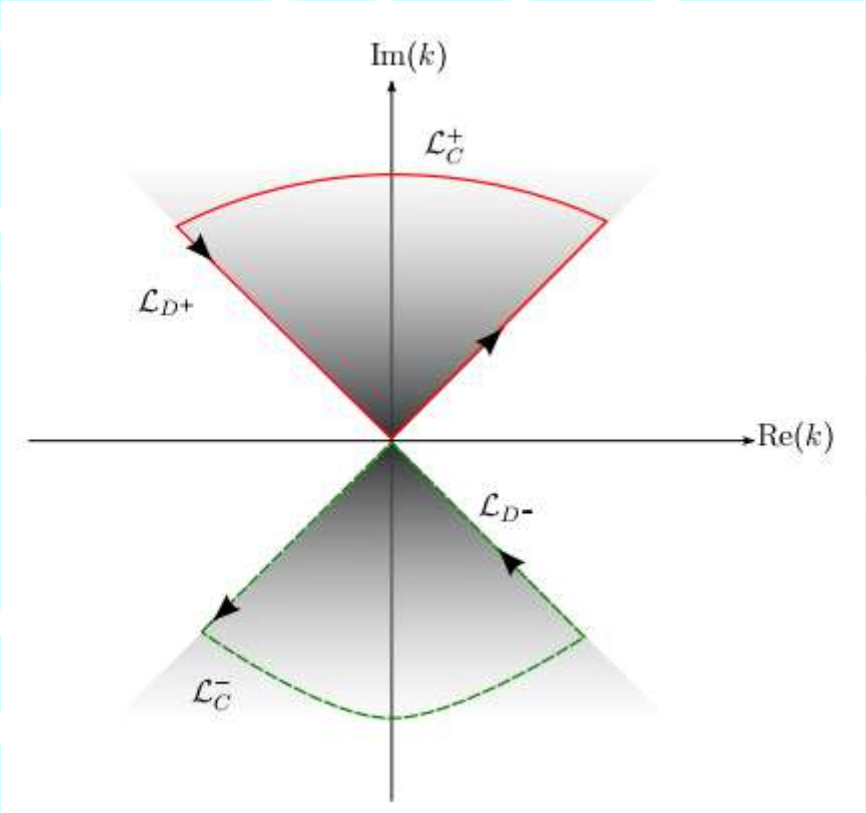
(9) tenglamada $g_1(w,t)$ noma'lum funksiya bor. Endi shu noma'lum funksiyani topish masalasini ko'rib chiqamiz. Buning uchun (7) tenglamadan $g_1(w,t)$ ni topamiz. Topilgan noma'lum funksiyani (9) ga qo'ysak, quyidagi integral ifoda hosil bo'ladi.

$$u(x,t)=\frac{1}{2p}\int_{-\Gamma}^{\Gamma} e^{ikx-k^2t}\hat{u}_0(k)dk-\frac{1}{p}\int_{\partial D^+} e^{ikx-k^2t}ikg_0(w,t)dk+$$

$$+\frac{1}{2p}\int_{\partial D^+} e^{ikx}\hat{u}(-k,t)dk-\frac{1}{2p}\int_{\partial D^+} e^{ikx-k^2t}\hat{u}_0(-k)dk. \qquad (10)$$

Bu ifoda o'ng tomonidagi uchinchi integralda biz izlayotgan yechimni o'z ichiga oladi. Biroq, $e^{ikx}\hat{u}(-k,t)$ yuqori yarim tekislikda golomorf va kamayuvchi funksiyadir. Shunday qilib, Jordan lemmasiga ko'ra J^+ dagi yopiq, chegaralangan egri chiziq bo'yicha $e^{ikx}\hat{u}(-k,t)$ funksiyadan olingan integral nolga intilishi kerak [8]. Demak, $L^+=L_{D^+}\cup L_{C^+}$ yopiq egri chiziqni ko'rib chiqamiz, bu yerda $L_{D^+}=\partial D^+\cap\{k:|k|<C\}$ va $L_{C^+}=\{k\in D^+:|k|=C\}$ (3-rasm).

ISSN 3030-3907



3-rasm.

Katta C uchun L_{C^+} integral nolga intilishi uchun (10) ning o‘ng tomonidagi uchinchi integral nolga intilishi kerak, chunki L_{D^+} kontur $C \to \Gamma$ da ∂D^+ ga o‘tadi. Jordan lemmasidan foydalansak, katta k uchun $\hat{u}(-k,t)$ ning tekis yemirilishi integralning nolga intilishi uchun zaruriy shart bo‘ladi. Biz masalani faqat boshlang‘ich va chegaraviy shartlar yordamida yechdik [7]:

$$u(x,t)=\frac{1}{2p}\int_{-\Gamma}^{\Gamma} e^{ikx-k^2t}\hat{u}_0(k)dk-\frac{1}{p}\int_{\partial D^+} e^{ikx-k^2t}ikg_0(w,t)dk-$$

$$-\frac{1}{2p}\int_{\partial D^+} e^{ikx-k^2t}\hat{u}_0(-k)dk,$$

bu yerda $g_0(w,t)$ oldin aytilgan Dirixle ma’lumotlarining vaqt o‘zgarishi.

Endi o‘zgarmas koeffitsiyentli chiziqli xususiy hosilali differensial tenglama uchun Fokas usulini ko‘rib chiqamiz

$$u_t(x,t)+w(-i\partial_x)u(x,t)=0, x>0, 0<t<T, \qquad (11)$$

bu yerda $w(k)$ n darajali ko`phad. $e^{ikx-w(k)t}$ yechimlar (2.11) tenglama bir oilani qabul qiladi. Yechimlar vaqt o‘tishi bilan eksponensial ravishda o‘sib bormasligini ta’minlash maqsadida haqiqiy k uchun $\mathrm{Re}(w(k))\geq 0$ bo‘lishini talab qilamiz.

ISSN 3030-3907

$$w(k)= \sum_{j=0}^{n} a_j k^j$$

bo‘lsin. Bu yerda $|k| \to \infty$ da $\mathrm{Re}(w(k)) \geq 0, k \in D$, agar n toq bo‘lsa $a_n = \pm i$ va agar n juft bo‘lsa $\mathrm{Re}(a_n) \geq 0$.

$D = \{k : \mathrm{Re}(w(k)) < 0\}$ sohalarni aniqlaymiz, $D^+ = D \cap \mathbb{C}^+$ va $D^- = D \cap \mathbb{C}^-$. Lokal munosabat quyidagicha aniqlanadi:

$$\partial_t \left(e^{-ikx+w(k)t} u(x,t)\right) - \partial_x \left(e^{-ikx+w(k)t} \sum_{j=0}^{n} c_j \partial_x^j u(x,t)\right) = 0,$$

bu yerda

$$\sum_{j=0}^{n} c_j(k) \partial_x^j u(x,t) = i \frac{w(k) - w(m)}{k - m}\Big|_{m=-i\partial_x} u(x,t),$$

va global munosabat esa

$$e^{w(k)T} \hat{u}(k,T) = \hat{u}_0(k) - \sum_{j=0}^{n-1} c_j(k) g_j(w(k),T),$$

bu yerda $\mathrm{Im}\, k \geq 0$ va

$$g_j(w(k),T) = \int_0^T e^{ws} \partial_x^j u(0,s)\, ds.$$

Global munosabatga teskari Fur’e almashtirishini qo‘llasak, yechimning integral ifodasini hosil qilamiz:

$$u(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx - w(k)t} \hat{u}_0(k)\, dk - \quad (12)$$

$$- \frac{1}{2\pi} \int_{\partial D^+} e^{ikx - w(k)t} \sum_{j=0}^{n} c_j(k) g_j(w(k),t)\, dk.$$

FOYDALANILGAN ADABIYOTLAR RO‘YXATI

1. Ablowitz M.J., Fokas A.S. Complex Variables: Introduction and Applications. // Cambridge University Press. New York.-2003.-P.660.
2. Berkolaiko G. An elementary introduction to quantum graphs // Contemporary Mathematics.-2017.Vol. 700.14182.-P.31.
3. Berkolaiko G., Kuchment P. Introduction to quantum graphs. //American Mathematical Soc.-2012.Vol. 186.-P. 290.
4. Fokas A.S. A unified transform method for solving linear and certain nonlinear PDEs // In Proc.R.Soc.A.-1997.Vol.453.-P.1411-1443.
6. Fokas A.S. A Unified Approach to Boundary Value Problems.Society for Industrial and Applied Mathematics.-2008.-P: 352.

ISSN 3030-3907

7. Fokas A.S., Pelloni B. A transform method for linear evolution PDEs on a finite interval // IMA Journal of Applied Mathematics.-2005.Vol. 70 (4).-P.564-587.
8. Fokas A. S., Pelloni B. Unified Transform for Boundary Value Problems // Application and Advances. SIAM. USA.-2015.-P.293.
9. Sheils N.E., Smith D. A. Heat equation on a network using the Fokas method. // Journal of Physics A:Mathematical and Theoretical.-2015.Vol. 48 (33).-P. 21.

ISSN 3030-3907